Contents

Chapter 1

INTRODUCTION: VECTOR ALGEBRA

1.1 Definitions

A **vector** may be defined in essentially three different ways: geometrically, analytically and axiomatically.

Definition 1.1.1. *Geometrically, a vector is defined as a collection of equivalent line segments.*

Equivalence, in this case, means that the line segments have the same *magnitude/length* and are *parallel*. Thus, a vector is characterised by its direction and magnitude/length.

Under this definition, the algebraic operations on vectors are introduced and studied geometrically, making maximum use of our geometric intuition.

Definition 1.1.2. *In the analytic approach, a vector in three-dimensional space is defined as an ordered triple of real numbers* (A_1, A_2, A_3) *relative to a given coordinate system. The real numbers* A_1, A_2, A_3 *are called the components of the vector.*

By far this approach is the most convenient for theoretical and computational considerations.

The axiomatic definition of a vector is left out as an exercise for you.

On the other hand, those quantities that are characterised by numerical magnitude alone, and have nothing to do with direction, are called **scalars** or **scalar quantities**. In order to distinguish vectors from scalars, we will use letters with an over-bar $\bar{A}, \bar{B}, \bar{C}, \cdots, \bar{a}, \bar{b}, \bar{c}, \cdots$, to denote vectors. In these typed notes I will be using boldfaced letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \cdots, \mathbf{a}, \mathbf{b}, \mathbf{c}, \cdots$.

In this course, we shall use mainly the analytic definition of a vector, occasionally giving the geometric interpretation of our results. Accordingly, we assume a (right-handed) coordinate system in our three-dimensional space. The coordinate system is said to be right-handed if the so called **Right-hand Rule** (to be demonstrated in class) holds.

Definition 1.1.3. *For any given (three-dimensional) vector* $\boldsymbol{A}$ *such that*

$$\boldsymbol{A} = (A_1, A_2, A_3),$$

A_1 *is called the first component,* A_2 *is called the second component,* A_3 *is called the third component of the vector* $\boldsymbol{A}$.

Definition 1.1.4. *A vector whose components are all zero is called the* ***zero vector*** *and is denoted by* $\boldsymbol{0}$ *or* $\bar{0}$, *that is,* $\boldsymbol{0} = (0, 0, 0)$.

Definition 1.1.5. *The* ***magnitude*** *of a vector* $\boldsymbol{A} = (A_1, A_2, A3)$, *denoted by* $|\boldsymbol{A}|$, *is the real number defined by*

$$|\boldsymbol{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{1.1}$$

Definition 1.1.6. *A vector whose magnitude is unit (or 1) is called a* ***unit*** *vector.*

For any vector $\mathbf{A}$, the corresponding unit vector, denoted $\hat{\mathbf{A}}$, is given by

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}. \tag{1.2}$$

From (1.2) we deduce that the vector $\mathbf{A}$ can be expressed as

$$\mathbf{A} = |\mathbf{A}|\hat{\mathbf{A}}. \tag{1.3}$$

1.2 Vector Algebra

Let us consider the Cartesian coordinate system in space, obtained by introducing three mutually perpendicular axes, labeled x, y, z, with the same unit of length along the three axes. The unit vectors in the positive x-, y- and z-directions are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, respectively. Thus, a vector $\mathbf{A}$ can be expressed in the form

$$\mathbf{A} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}. \tag{1.4}$$

The numbers A_1, A_2 and A_3 are called the **orthogonal projections** or **components** of $\mathbf{A}$ in the x-, y- and z-directions, respectively.

Two vectors $\mathbf{A} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$ and $\mathbf{B} = B_1\hat{\mathbf{i}} + B_2\hat{\mathbf{j}} + B_3\hat{\mathbf{k}}$ are said to be equal if

$$A_1 = B_1, \qquad A_2 = B_2, \qquad A_3 = B_3.$$

Exercise: If $|\mathbf{A}| = |\mathbf{B}|$, is it necessarily true that $\mathbf{A} = \mathbf{B}$?

1.2.1 Addition of vectors

Vector addition proceeds componentwise, that is

$$\mathbf{A} + \mathbf{B} = (A_1 + B_1)\hat{\mathbf{i}} + (A_2 + B_2)\hat{\mathbf{j}} + (A_3 + B_3)\hat{\mathbf{k}}.$$

It follows that addition of vectors satisfies the commutative law

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A},$$

and the associative law

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C},$$

for any vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

1.2.2 Scalar multiplication of vectors

$$\lambda\mathbf{A} = \lambda(A_1)\hat{\mathbf{i}} + \lambda(A_2)\hat{\mathbf{j}} + \lambda(A_3)\hat{\mathbf{k}},$$

where λ is a scalar.

Exercise: Verify that for any vectors $\mathbf{A}, \mathbf{B}$ and for any scalars m, n the following hold:

1. $m(n\mathbf{A}) = (mn)\mathbf{A}$,

2. $m(\mathbf{A}+\mathbf{B}) = m\mathbf{A} + m\mathbf{B}$,
3. $(m+n)\mathbf{A} = m\mathbf{A} + n\mathbf{A}$,
4. $1\mathbf{A} = \mathbf{A}$,
5. $0\mathbf{A} = \mathbf{0}$.

An alternatively description of a vector in space is obtained by specifying its magnitude and direction. The direction can be specified by prescribing the three direction angles α, β and γ between the vector and the positive x, y and z directions respectively. The magnitude is determined using the Pythagorean Theorem as

$$|\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}.$$

Exercise: Verify the following formulas, for a vector $\mathbf{A} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$:

$$\cos\alpha = \frac{A_1}{|\mathbf{A}|}, \qquad \cos\beta = \frac{A_2}{|\mathbf{A}|}, \qquad \cos\gamma = \frac{A_3}{|\mathbf{A}|},$$

and hence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

When the initial point of a vector is fixed, it is called a **fixed** or **localised** vector, otherwise it is a **free** vector.

1.2.3 The scalar product

The **scalar product** (also known as the dot or inner product) of two vectors $\mathbf{A}, \mathbf{B}$ is the number

$$\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3. \tag{1.5}$$

Alternatively,

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta, \tag{1.6}$$

where θ is the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$. We can interpret $\mathbf{A} \cdot \mathbf{B}$ as

$$(\text{length of} \quad \mathbf{A}) \times (\text{signed component of} \quad \mathbf{B} \quad \text{along} \quad \mathbf{A}).$$

Since the definition is symmetric in $\mathbf{B}$ and $\mathbf{B}$, it can equally be interpreted as

$$(\text{length of} \quad \mathbf{B}) \times (\text{signed component of} \quad \mathbf{A} \quad \text{along} \quad \mathbf{B}).$$

Application: If $\mathbf{F}$ is a constant force acting through a displacement $\mathbf{d}$, the work done by $\mathbf{F}$ is defined as the product of the magnitude of the force and the component of the displacement in the direction of the force, that is,

$$\text{Work} \quad \text{done} = \mathbf{F} \cdot \mathbf{d}.$$

The following properties of the scalar product can easily be verified:

1. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$,
2. $(\lambda \mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \lambda \mathbf{A} \cdot \mathbf{C} + \lambda \mathbf{B} \cdot \mathbf{C}$,
3. $|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A}$.

Exercise 1.2.1. *1. Find the component of the vector $8\hat{\mathbf{i}} + \hat{\mathbf{j}}$ in the direction of the vector $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.*

2. Find the vector in the same direction as $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ whose component in the direction of $2\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$ is unit.

1.2.4 The vector product

The **vector product** (also known as the cross product) of two vectors $\mathbf{A}, \mathbf{B}$ is defined by

$$\mathbf{A} \times \mathbf{B} = (A_2 B_3 - A_3 B_2)\hat{\mathbf{i}} + (A_3 B_1 - A_1 B_3)\hat{\mathbf{j}} + (A_1 B_2 - A_2 B_1)\hat{\mathbf{k}}. \tag{1.7}$$

A more convenient formula for the vector product which makes use of the notion of a determinant of a matrix is as follows:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}. \tag{1.8}$$

Exercises: By applying the definition of a vector product, verify the following

1. $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}$,
2. $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$,
3. $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$,

for unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

Geometrically, the vector product vectors $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \hat{\mathbf{n}} \tag{1.9}$$

where θ is the angle between the vectors $\mathbf{A}, \mathbf{B}$ and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. Thus,

$$\mathbf{A} \times \mathbf{B}$$

is a vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. We further observe that the magnitude of the vector $\mathbf{A} \times \mathbf{B}$,

$$|\mathbf{A} \times \mathbf{B}| = ||\mathbf{A}||\mathbf{B}| \sin\theta \hat{\mathbf{n}}| = |\mathbf{A}||\mathbf{B}|| \sin\theta|,$$

which is the area of a parallelogram whose edges are the vectors $\mathbf{A}$ and $\mathbf{B}$.
The following properties of the vector product can easily be verified:

1. $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$,
2. $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$,
3. $\mathbf{A} \times \mathbf{A} = \mathbf{0}$.

Equations of lines

The **position vector of a point** (x, y, z) is the vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Geometrically, the position vector of a point is the directed line extending from the origin to the point.

The equation of a line is completely determined if we know any two points on the line or if we know a point on the line and the orientation or direction of the line. The direction of the line can be described by a vector to which the line is parallel. Suppose we wish to find the equation of a line that passes through a given point (x_0, y_0, z_0) and is parallel to a given non-zero vector $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$. Let $\mathbf{r}$ denote the position vector of an arbitrary point (x, y, z) on the line. if $\mathbf{r}_0 = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} + z_0\hat{\mathbf{k}}$ denotes the position vector of the point (x_0, y_0, z_0), then the vector $\mathbf{r} - \mathbf{r_0}$ is parallel to the vector $\mathbf{v}$. Hence there exists a scalar t such that $\mathbf{r} - \mathbf{r_0} = t\mathbf{v}$. Thus the position vector $\mathbf{r}$ of an arbitrary point on the line is given by

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \qquad t_1 \leq t \leq t_2. \tag{1.10}$$

this is a vector equation of the line. The scalar t is called the **parameter**; and as the parameter ranges from t_1 to t_2, the vector $\mathbf{r}$ traces the line from one end to the other with $t = 0$ corresponding to the point (x_0, y_0, z_0). Equating the corresponding components of the vectors in (1.10) we obtain the **parametric equations** of the line:

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + tv_3. \tag{1.11}$$

If we eliminate the parameter t from the equations (1.11), we obtain the symmetric form of the equation of the line:

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}. \tag{1.12}$$

Example 1.2.1. *1. Find a vector equation of the line passing through the points* $(1, -2, 1)$ *and* $(3, 1, 1)$.

2. Deduce the corresponding parametric equations and symmetric equations.

Solution 1.2.1. *1. We note that the line is parallel to the vector*

$$\mathbf{v} = (3 - 1)\hat{\mathbf{i}} + (1 - (-2))\hat{\mathbf{j}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}.$$

Let $\mathbf{r_0} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, *then*

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v} = (1 + 2t)\hat{\mathbf{i}} + (3t - 2)\hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

2. By equating the corresponding components of the vector equation, we obtain the parametric equations

$$x = 1 + 2t, \qquad y = -2 + 3t, \qquad z = 1.$$

By eliminating the parameter t*, we obtain the equation in symmetric form*

$$\frac{x - 1}{2} = \frac{y + 2}{3}, z = 1.$$

Exercise 1.2.2. *1. Find the parametric equations of the line that passes through the point* $(-1, 3, -2)$ *and is perpendicular to the vectors* $\mathbf{A} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ *and* $\mathbf{A} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

2. Derive the symmetric form (1.12) of the equation of the line from the equation

$$(\mathbf{r} - \mathbf{r_0}) \times \mathbf{v} = \mathbf{0}. \tag{1.13}$$

3. Make notes on the derivation of equations of planes.

1.2.5 The scalar triple product

The **scalar triple product** of vectors **A**, **B** and **C**, denoted $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$, is defined by

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}). \tag{1.14}$$

In terms of the components of the vectors

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = A_1(B_2C_2 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1). \tag{1.15}$$

It is interesting to note that the scalar triple product of three non-zero vectors represents, up to sign, the volume of the parallelepiped formed by the vectors **A**, **B** and **C**.

Example 1.2.2. *Let P:* $(1, -2, 3)$*, Q:* $(2, 1, -2)$*, R:* $(-2, 1, -1)$ *and S:* $(2, 2, 3)$ *be four given points. Find the volume of the parallelepiped formed by the vectors* $\mathbf{A} = \overline{PQ}$*,* $\mathbf{B} = \overline{PR}$ *and* $\mathbf{C} = \overline{PS}$*.*

Solution 1.2.2.

$$\mathbf{A} = \hat{\mathbf{i}} + +\hat{\mathbf{j}} - 5\hat{\mathbf{k}}, \qquad \mathbf{B} = -3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}, \qquad \mathbf{C} = \hat{\mathbf{i}} + 4\hat{\mathbf{j}}.$$

Thus, the volume of the parallelepiped is

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \begin{vmatrix} 1 & 3 & -5 \\ -3 & 3 & -4 \\ 1 & 4 & 0 \end{vmatrix} = 79.$$

Exercise 1.2.3. *Evaluate* $[\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}]$.

1.2.6 The vector triple product

The **vector triple product** of vectors **A**, **B** and **C** is defined by

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{B} \cdot \mathbf{A})\mathbf{C}. \tag{1.16}$$

Exercise: Prove the identity

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}|^2 |\mathbf{B}|^2 - (\mathbf{A} \cdot \mathbf{B})^2.$$

Chapter 2

DIFFERENTIATION OF VECTOR FUNCTIONS

2.1 The Vector Function

Definition 2.1.1. *A* **vector function** *or* **vector-valued function** $\mathbf{F}(t)$ *is a mathematical rule that associates a vector* $\mathbf{F}$ *with each real number* t *in some set, usually an interval* $t_1 \leq t \leq t_2$ *or a collection of intervals* $(t_1 \leq t \leq t_2) \cup (t_3 \leq t \leq t_4) \cup \cdots \cup (t_{n-1} \leq t \leq t_n)$.

In the Cartesian coordinate system, the vector function can be expressed in component form, say

$$\mathbf{F}(t) = F_1(t)\hat{\mathbf{i}} + F_2(t)\hat{\mathbf{j}} + F_3(t)\hat{\mathbf{k}} \tag{2.1}$$

where F_1, F_2 and F_3 are scalar-valued functions of t and are called the components of $\mathbf{F}(t)$. For example, the equation

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}, \qquad t_1 \leq t \leq t_2,$$

of a line that passes through the point $\mathbf{r_0}$ and is parallel to the vector $\mathbf{v}$ is a vector function of the parameter t.

2.1.1 Limit of a vector function

Definition 2.1.2. *Let* $\mathbf{F}(t)$ *be a vector function defined in an interval* $a \leq t \leq b$ *(*$t = t_0$ *is a point in this interval) and* $\mathbf{A}$ *a constant vector. We say that the* **limit** *of* $\mathbf{F}(t)$ *as* t *approaches* t_0 *is* $\mathbf{A}$*, written*

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{A} \tag{2.2}$$

if and only if for any given real number $\epsilon > 0$*, there is a real number* $\delta > 0$ *such that*

$$\mathbf{F}(t) - \mathbf{A}| < \epsilon \qquad \text{whenever} \qquad 0 < |t - t_0| < \delta. \tag{2.3}$$

This definition means that the magnitude of $\mathbf{F}(t)$ is approaching the magnitude of $\mathbf{A}$ and that the angle between them is approaching zero, provided $\mathbf{A} \neq \mathbf{0}$.

Theorem 2.1.1. *If* $\mathbf{F}(t) = F_1(t)\hat{\mathbf{i}} + F_2(t)\hat{\mathbf{j}} + F_3(t)\hat{\mathbf{k}}$ *and* $\mathbf{A} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$*, then* $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{A}$ *if and only if* $\lim_{t\to t_0} F_i(t) = A_i, \quad i = 1, 2, 3.$

2.1.2 Continuity of a vector function

Definition 2.1.3. *A vector valued function* $\mathbf{F}(t)$ *is said to be* **continuous** *at a point* $t = t_0$ *if and only if*

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{F}(t_0). \tag{2.4}$$

Therefore, $\mathbf{F}(t)$ *is continuous at* $t = t_0$ *if and only if, for a given* $\epsilon > o$*, there exists* $\delta > 0$ *such that*

$$|\mathbf{F}(t) - \mathbf{F}(t_0)| < \epsilon, \qquad \text{whenever} \qquad |t - t_0| < \delta.$$

If (2.4) holds for all points in its interval of definition, then $\mathbf{F}$ is said to be continuous in that interval.

Theorem 2.1.2. *If* $\mathbf{F}(t) = F_1(t)\hat{\mathbf{i}} + F_2(t)\hat{\mathbf{j}} + F_3(t)\hat{\mathbf{k}}$ *and* $\mathbf{A} = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$*, then*

$$\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{A} \quad if \;\; and \;\; only \;\; if \quad \lim_{t\to t_0} F_i(t) = A_i, \qquad i = 1, 2, 3.$$

Theorem 2.1.3. *If* $\lim_{t\to t_0} \mathbf{F}(t) = \mathbf{A}$ *and* $\lim_{t\to t_0} \mathbf{G}(t) = \mathbf{B}$*, then*

$$\lim_{t\to t_0} [\mathbf{F}(t) + \mathbf{G}(t)] = \mathbf{A} + \mathbf{B}.$$

Theorem 2.1.4. *A vector function* $\mathbf{F}(t) = F_1(t)\hat{\mathbf{i}} + F_2(t)\hat{\mathbf{j}} + F_3(t)\hat{\mathbf{k}}$ *is continuous at a point* $t = t_0$ *if each* $F_i(t)$ $(i - 1, 2, 3)$ *is continuous at* $t = t_0$*.*

Exercise 2.1.1. *Prove the above theorems.*

2.1.3 Derivative of a vector function

Definition 2.1.4. *The* **derivative** *of a vector valued function* $\mathbf{F}(t)$ *at a point* $t = t_0$*, denoted by* $\mathbf{F}'(t_0) = \frac{d\mathbf{F}(t)}{dt}$*, is defined as the limit*

$$\mathbf{F}'(t_0) = \lim_{t\to \Delta t_0} \frac{\mathbf{F}(t_0 + \Delta t) - \mathbf{F}(t_0)}{\Delta t}, \tag{2.5}$$

provided the limit exists.

In terms of components, if $\mathbf{F}(t) = F_1(t)\hat{\mathbf{i}} + F_2(t)\hat{\mathbf{j}} + F_3(t)\hat{\mathbf{k}}$, then (2.5) may be written as

$$\mathbf{F}'(t) = \lim_{\Delta t\to 0} \left[\frac{F_1(t_0 + \Delta t) - F_1(t_0)}{\Delta t}\hat{\mathbf{i}} + \frac{F_2(t_0 + \Delta t) - F_2(t_0)}{\Delta t}\hat{\mathbf{j}} + \frac{F_3(t_0 + \Delta t) - F_3(t_0)}{\Delta t}\hat{\mathbf{k}} \right].$$

Therefore, if the components $F_1(t), F_2(t), F_3(t)$ are differentiable, then we have

$$\mathbf{F}'(t) = F_1'(t)\hat{\mathbf{i}} + F_2'(t)\hat{\mathbf{j}} + F_3'(t)\hat{\mathbf{k}}. \tag{2.6}$$

Thus, a vector function $\mathbf{F}$ is differentiable in an interval if and only if its components are all differentiable in that interval. Further, observe that:

1. The derivative of a vector-valued function is also a vector-valued function.
2. $\mathbf{F}(t)$ is continuous in its interval of definition whenever $\mathbf{F}'(t)$ exists in that interval.

Theorem 2.1.5. *If* $\mathbf{F}(t)$ *and* $\mathbf{G}(t)$ *are differentiable vector functions, then so is their sum* $\mathbf{F}(t) + \mathbf{G}(t)$*, and*

$$\frac{d}{dt}(\mathbf{F} + \mathbf{G}) = \frac{d\mathbf{F}}{dt} + \frac{d\mathbf{G}}{dt}. \tag{2.7}$$

Theorem 2.1.6. *If $\mathbf{F}(t)$ is a differentiable vector function, and $\phi(t)$ is a differentiable scalar function, then the product $\phi\mathbf{F}$ is a differentiable vector function, and*

$$\frac{d}{dt}(\phi\mathbf{F}) = \frac{d\phi}{dt}\mathbf{F} + \phi\frac{d\mathbf{F}}{dt}. \tag{2.8}$$

Theorem 2.1.7. *If $\mathbf{F}(t)$ and $\mathbf{G}(t)$ are differentiable vector functions, then $\mathbf{F}\cdot\mathbf{G}$ is a differentiable scalar function, and*

$$\frac{d}{dt}(\mathbf{F}\cdot\mathbf{G}) = \frac{d\mathbf{F}}{dt}\cdot\mathbf{G} + \mathbf{F}\cdot\frac{d\mathbf{G}}{dt}. \tag{2.9}$$

Theorem 2.1.8. *If $\mathbf{F}(t)$ and $\mathbf{G}(t)$ are differentiable vector functions, then $\mathbf{F}\times\mathbf{G}$ is a differentiable vector function, and*

$$\frac{d}{dt}(\mathbf{F}\times\mathbf{G}) = \frac{d\mathbf{F}}{dt}\times\mathbf{G} + \mathbf{F}\times\frac{d\mathbf{G}}{dt}. \tag{2.10}$$

Note that when differentiating a vector product of two vector functions, one must be careful to preserve the order of factors, since the cross product of vectors is not a commutative operation.

Exercise 2.1.2. *Prove the above theorems.*

2.1.4 The partial derivative of a vector function

Consider a vector function $\mathbf{F}$ expressed in terms of the Cartesian coordinate system as

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}. \tag{2.11}$$

Definition 2.1.5. *The partial* **derivative** *of the vector function $\mathbf{F}(x,y,z)$ with respect to x,y,z, denoted $\mathbf{F}_x = \frac{\partial\mathbf{F}}{\partial x}$, $\mathbf{F}_y = \frac{\partial\mathbf{F}}{\partial y}$, $\mathbf{F}_z = \frac{\partial\mathbf{F}}{\partial z}$ respectively, is defined as the limit*

$$\mathbf{F}_x = \lim_{\Delta x\to 0}\frac{\mathbf{F}(x+\delta x,y,z) - \mathbf{F}(x,y,z)}{\Delta x}, \tag{2.12}$$

$$\mathbf{F}_y = \lim_{\Delta y\to 0}\frac{\mathbf{F}(x,y+\delta y,z) - \mathbf{F}(x,y,z)}{\Delta y}, \tag{2.13}$$

$$\mathbf{F}_z = \lim_{\Delta z\to 0}\frac{\mathbf{F}(x,y,z+\delta z) - \mathbf{F}(x,y,z)}{\Delta z} \tag{2.14}$$

provided the limits exist.

Exercise 2.1.3. *From the definition of a partial derivative, show that*

$$\frac{\partial\mathbf{F}}{\partial x} = \frac{\partial F_1}{\partial x}\hat{\mathbf{i}} + \frac{\partial F_2}{\partial x}\hat{\mathbf{j}} + \frac{\partial F_3}{\partial x}\hat{\mathbf{k}}.$$

2.2 Space Curves and Tangent Vectors

A curve in space may be defined as a set of points (x, y, z) determined by three equations of the form

$$x = x(t), \quad y = y(t), \quad z = z(t), \tag{2.15}$$

where the functions $x(t)$, $y(t)$, $z(t)$ are assumed to be continuous functions of t in some interval $t_1 \leq t \leq t_2$. The equations (2.15) are called parametric equations of the curve and t is called the parameter. To obtain a vector equation of the curve, we consider the position $\mathbf{r}(t)$ of each point on the curve corresponding to the parameter t. since the components of $\mathbf{r}(t)$ are precisely the coordinates of the point, it follows that

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}, \quad (t_1 \leq t \leq t_2). \tag{2.16}$$

Consider the following examples of special space curves:

1. The parametric equation of a straight line is written in vector form as:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad (t_1 \leq t \leq t_2). \tag{2.17}$$

   where $\mathbf{r}_0$ is the position vector of a fixed point on the line, $\mathbf{v}$ is a vector parallel to the line and t is a parameter such that as t assumes values in the interval $(t_1 \leq t \leq t_2)$, the tip of the vector $\mathbf{r}$ traces out the straight line in space.

2. The vector function

$$\mathbf{r}(t) = (1 + a\cos t)\hat{\mathbf{i}} + (1 + a\sin t)\hat{\mathbf{j}}, \quad (0 \leq t \leq 2\pi), \tag{2.18}$$

   represents a circle of radius a and center at the point $(1, 1)$. The corresponding parametric equations of the circle described above are:

$$x = 1 + a\cos t, \qquad y = 1 + a\sin t \qquad (0 \leq t \leq 2\pi). \tag{2.19}$$

   By eliminating the parameter t from the parametric equations we obtain

$$(x-1)^2 + (y-1)^2 = a^2 \tag{2.20}$$

   which is the standard equation of the circle in Cartesian coordinates.

2.2.1 Orientation of space curves

A curve that is represented in parametric or vector equation can be given one of the two possible directions with respect to the parameter t in a natural way. The **positive direction** on the curve is the direction in which the curve is traced as the parameter t increases from t_1 to t_2. The opposite direction is called the **negative direction**.

A curve on which direction has been prescribed is said to be **oriented**.

The parametric representation of of a space curve is not unique. This means that it is possible, if we so desire, to change the parameter t in equation (2.15) or (2.16) to another parameter, say s, by setting $t = f(s)$ where f is any differentiable scalar function such that $f' \neq 0$, without changing the curve itself.

2.2.2 Tangent vector on a space curve

Let C be a space curve represented represented by the equation

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}, (t_1 \leq t \leq t_2), \tag{2.21}$$

where the functions $x(t)$, $y(t)$, $z(t)$ are continuously differentiable in $t_1 \leq t \leq t_2$. By definition,

$$\frac{d}{dt}\mathbf{r}(t) = \lim_{\Delta t} \rightarrow 0 \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}. \tag{2.22}$$

We call the vector represented by $\frac{d\mathbf{r}(t)}{dt}$ a tangent vector to the curve at the point corresponding a value of the parameter t. The vector

$$\hat{\mathbf{T}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \tag{2.23}$$

is called the **unit tangent vector**. In terms of its components, this unit tangent vector is given by

$$\hat{\mathbf{T}} = \frac{x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}}}{\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}}. \tag{2.24}$$

For example, the curve defined by the vector equation

$$\mathbf{r}(t) = a\cos t\hat{\mathbf{i}} + a\sin t\hat{\mathbf{j}} + bt\hat{\mathbf{k}}, \quad (t \geq 0),$$

where a and b are positive constants, can be expressed as $x^2 + y^2 = a^2$. We see that the curve lies on the lateral surface of a circular cylinder of radius a whose axis is the z-axis. Moreover, as t increases the coordinate z increases since $b > 0$. Therefore, as t increases, the curve spirals upward around the cylinder in the counterclockwise direction as view from atop the positive z-axis. The curve is called a *circular helix.*

a tangent vector to the curve for any value of t ($t > 0$) is given by

$$\mathbf{r}'(t) = -a \sin t\hat{\mathbf{i}} + a \cos t\hat{\mathbf{j}} + b\hat{\mathbf{k}}.$$

The unit tangent vector is

$$\hat{\mathbf{T}} = \frac{a(-\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}}) + b\hat{\mathbf{k}}}{\sqrt{a^2 + b^2}}.$$

Exercise 2.2.1. *Find an equation of the tangent line to the curve*

$$\mathbf{r}(t) = e^t \cos t\hat{\mathbf{i}} + e^t \sin t\hat{\mathbf{j}}, t \geq 0$$

at the point corresponding to $t = \frac{\pi}{4}$.

Definition 2.2.1. *A space curve is said to be* **smooth** *if it has a parametrization* $\mathbf{r}(t)$, $t_1 \leq t \leq t_2$, *satisfying the following conditions:*

1. $\frac{d\mathbf{r}}{dt}$ *exists and is a continuous function of* t*, for all values of* t *in the interval* $t_1 \leq t \leq t_2$.
2. *to distinct values of* t *in the interval* $t_1 \leq t \leq t_2$ *there corresponds distinct points* $\mathbf{r}(t)$.
3. *there is no value of* t *in the interval* $t_1 \leq t \leq t_2$ *for which* $\frac{d\mathbf{r}}{dt}$ *is the zero vector.*

Examples of smooth curves are: lines, circles, parabolas, spirals, helices, etc.

Definition 2.2.2. *We say that a space curve is* **piecewise smooth** *if* $\mathbf{r}(t)$ *has piecewise continuous non-zero derivatives in* $t_1 \leq t \leq t_2$. *For example, the sides of a rectangle constitute a piecewise smooth curve.*

Definition 2.2.3. *A space curve* $\mathbf{r}(t)$, $t_1 \leq t \leq t_2$, *is said to be* **closed** *if* $\mathbf{r}(t_1) = \mathbf{r}(t_2)$. *If, in addition, each point on the curve corresponds to one and only one value of the parameter* t*, other than* $t = t_1$ *and* $t = t_2$*, then the curve is called a* **simple closed curve**.

2.2.3 Arc length

Let C be a smooth curve represented by $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + x(t)\hat{\mathbf{k}}$, $t_1 \leq t \leq t_2$. The **arc length**, denoted by $s(t)$, of the curve from the point corresponding to $t = t_1$ to a point corresponding to an arbitrary value of $t \in [t_1, t_2]$ is given by the integral

$$s(t) = \int_{t_1}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2} d\tau. \tag{2.25}$$

Example 2.2.1. *Find the arc length of the curve represented by the equation*

$$\mathbf{r}(t) = e^t(\cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}}), \qquad 0 \leq t \leq \pi.$$

Solution
We have

$$\mathbf{r}'(t) = e^t(\cos t - \sin t)\hat{\mathbf{i}} - e^t(\sin t + \cos t)\hat{\mathbf{j}}$$

so that

$$|\mathbf{r}(t)| = \sqrt{2}e^t.$$

Hence by (2.26) the arc length of the curve is

$$s(t) = \int_0^{\pi} \sqrt{2}e^t dt = \sqrt{2}(e^{\pi} - 1).$$

Exercise 2.2.2. *Find the arc length of*

1. *the curve*
$$x = t, \qquad y = 2t + 5, \qquad , z = 3t$$
between $(0, 5, 0)$ *and* $(1, 7, 3)$.

2. *the curve*
$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad , z = 0$$
between $t = 0$ *and* $t = 1$.

3. *the helix winding about the* x*-axis*
$$y = \sin 2\pi x, \qquad , z = \cos 2\pi x$$
between $(0, 0, 1)$ *and* $(1, 0, 1)$.

Since the integral in (2.25) is precisely the magnitude of $\frac{d\mathbf{r}(t)}{dt}$, we can write (2.25) as

$$s(t) = \int_{t_1}^{t} \left| \frac{d\mathbf{r}(\tau)}{d\tau} \right| d\tau. \tag{2.26}$$

Thus,

$$\frac{ds}{dt}(t) = \left| \frac{d\mathbf{r}}{dt}(t) \right|. \tag{2.27}$$

Equation (2.27) implies that

$$\left(\frac{ds}{dt} \right)^2 = \left| \frac{d\mathbf{r}}{dt} \right|^2 = \frac{d\mathbf{r}}{dt} \cdot dt. \tag{2.28}$$

Hence, if we define the vector differential $d\mathbf{r}$ by the equation

$$d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \tag{2.29}$$

then we have

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (dx)^2 + (dy)^2 + (dz)^2. \tag{2.30}$$

The differential ds is called the **element of arc length**.

Exercise 2.2.3. 1. *Find the length of the curve defined by*

$$\mathbf{r}(t) = \sin t\hat{\mathbf{i}} + t\hat{\mathbf{j}} + (1 - \cos t)\hat{\mathbf{k}}$$

for $0 \leq t \leq 2\pi$.

2. *Find the length of the circular helix*

$$\mathbf{r}(t) = a(\cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}}) + bt\hat{\mathbf{k}}$$

from 0 *to* 2π.

3. *Let* C *be a curve in the* xy*-plane represented in polar coordinates by the equation*

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad r = f(\theta),$$

where f *is a continuous differentiable function. Show that*

$$(ds)^2 = [f^2(\theta) + f'^2(\theta)](d\theta)^2.$$

Chapter 3

DIFFERENTIAL CALCULUS OF SCALAR AND VECTOR FIELDS

3.1 Scalar Fields

If to each point (x, y, z) of a region, D, in space there is made to correspond a number $f(x, y, z)$, we say that f is a **scalar field** in the domain D.

3.1.1 Examples of scalar fields

1. The water pressure at each point in a sea.
2. $f(x, y, z) = x + 2y - z$.
3. $\phi(r, s, t) = \frac{r^2}{4} + \frac{s^2}{9} + t^2$.
4. The density of atmospheric air in a room.
5. The electrostatic potential of the region between two condenser plates.

If $f(x, y, z)$ is a scalar field, any surface defined by

$$f(x, y, z) = C, \tag{3.1}$$

where C is a constant, is called an **isotimic surface**. In other words, an isotimic surface is a locus of points at which the scalar function of the field assumes the same value.

Sometimes more specialised terms are used in place of "isotimic surface":

1. If $f(x, y, z)$ denotes pressure, such surfaces are called *isobaric surfaces.*
2. If $f(x, y, z)$ denotes temperature, the surfaces are called *isothermal surfaces*
3. If $f(x, y, z)$ denotes electric or gravitational field potential, they are called *equipotential surfaces*

Let us consider the behavior of a scalar field in the neighborhood of a point (x_0, y_0, z_0) within its domain of definition. In many applications, it is often necessary to know the rate of change of f in an arbitrary direction.

Definition 3.1.1. *The derivative of f at (x_0, y_0, z_0), denoted $\frac{df}{ds}$, where s is measured in the direction of a vector* **u***, if it exists, is called the* **directional derivative** *of f at (x_0, y_0, z_0) in the direction of a vector* **u***.*

In other words, the directional derivative of f is simply the rate of change of f, per unit distance in some prescribed direction.

Note that the directional derivative $\frac{df}{ds}$ will generally depend on the location of the point (x_0, y_0, z_0) and also on the prescribed direction **u**.

For a scalar field f, $\frac{df}{ds}$ in the direction parallel to the x-axis with s measured as increasing in the positive x-direction is conveniently denoted $\frac{\partial f}{\partial x}$ and is called the partial derivative of f with respect to x. Similarly, $\frac{df}{ds}$ in the direction parallel to the y-axis with s measured as increasing in the positive y-direction is denoted $\frac{\partial f}{\partial y}$ and is called the partial derivative of f with respect to y; and $\frac{df}{ds}$ in the direction parallel to the z-axis with s measured as increasing in the positive z-direction is denoted $\frac{\partial f}{\partial z}$ and is called the partial derivative of f with respect to z.

We define the directional derivative of f in a direction parallel to a vector **u** by

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \tag{3.2}$$

provided the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ exist and are continuous throughout the region of definition.

Observe that

$$\hat{\mathbf{u}} = \frac{dx}{ds}\hat{\mathbf{i}} + \frac{dy}{ds}\hat{\mathbf{i}} + \frac{dz}{ds}\hat{\mathbf{k}} \tag{3.3}$$

is a unit vector pointing in the direction in which s is measured. Hence, by defining the **gradient** of f to be the vector

$$\text{grad} f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{i}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \tag{3.4}$$

we deduce that the right-hand side of (3.2) is the scalar product of grad$f$ and $\hat{\mathbf{u}}$, thus

$$\frac{df}{ds} = \text{grad} f \cdot \mathbf{u}. \tag{3.5}$$

The gradient of a scalar field f is also commonly written as ∇f, where ∇ (known as the **del** operator) denotes the vector differential operator

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}. \tag{3.6}$$

In other words, grad f can be interpreted as the result of applying the differential operator ∇ to f.

Exercise 3.1.1. *Prove that $\nabla(fg) = f\nabla g + g\nabla f$, for sufficiently continuous scalar fields f and g.*

3.1.2 Some properties of the gradient

By definition the scalar product

$$\nabla f \cdot \hat{\mathbf{u}} = |\nabla f||\hat{\mathbf{u}}|\cos\theta,$$

where θ is the angle between ∇f and the unit vector $\hat{\mathbf{u}}$. Thus

$$\frac{df}{ds} = |\nabla f|\cos\theta. \tag{3.7}$$

We deduce, from (3.7), that

1. The directional derivative of a scalar field f at a point (x_0, y_0, z_0) in the direction of the unit vector $\hat{\mathbf{u}}$ is simply the signed component of the gradient vector ∇f at (x_0, y_0, z_0) along $\hat{\mathbf{u}}$.
2. The directional derivative is maximum when $\cos\theta = 1$, that is, when $\hat{\mathbf{u}}$ is in the same direction as ∇f.

3. The maximum possible value of $\frac{df}{ds}$ is given by $|\nabla f|$.

4. Through any point (x, y, z) where $\nabla f \neq 0$, there passes an isotimic surface $f(x, y, z) = c$, such that ∇f is normal/orthogonal to this surface at the point (x, y, z). The vector

$$\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|}$$

is then a unit normal vector to the surface.

Example 3.1.1. *Find the directional derivative of the scalar field*

$$f(x, y, z) = x^2 + y^2 - z$$

in the direction of the vector $4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$*, at the point* $(1, 1, 2)$*.*

Solution 3.1.1.

$$\frac{df}{ds} = \nabla f \cdot \hat{\mathbf{u}}.$$

Now

$$\nabla f = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - \hat{\mathbf{k}} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

at $(1, 1, 2)$*. A unit vector in the desired direction is*

$$\hat{\mathbf{u}} = \frac{1}{3}(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}).$$

Thus

$$\frac{df}{ds} = (2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) \cdot (\frac{2}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} - \frac{1}{3}\hat{\mathbf{k}}) = 3.$$

This means that the value of the scalar field f is increasing 3 units/unit distance, if we proceed from $(1, 1, 2)$ in the stated direction.

Example 3.1.2. *The temperature of points in space is given by*

$$f(x, y, z) = x^2 + y^2 - z.$$

1. *A mosquito located at* $(1, 1, 2)$ *desires to fly in such a direction that it will get cool as soon as possible. In what direction should it fly?*

2. *Another mosquito is flying at a speed of 5 units of distance per second, in the direction of the vector* $4\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$*. What is its rate of increase of temperature, per unit time, at the instant it passes through the point* $(1, 1, 2)$*?*

Solution 3.1.2. 1. *To get cool as soon as possible, the mosquito should fly in the direction*

$$-\nabla f(1,1,2) = -2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}},$$

from example (3.1.1).

2. *The rate of increase of temperature per unit distance is*

$$\frac{df}{ds} = 3,$$

from example (3.1.1). The speed of flight of the mosquito is $\frac{ds}{dt} = 5$*. Hence the rate of increase of temperature per unit time will be given, using the chain rule, by*

$$\frac{df}{dt} = \frac{df}{ds}\frac{ds}{dt} = 3 \times 5 = 15\text{units/second}.$$

Example 3.1.3. *Find a unit vector normal to the surface*

$$x^2 + y^2 - z = 6$$

at the point $(2,3,7)$*.*

Solution 3.1.3. *The surface* $x^2 + y^2 - z = 6$ *is an isotimic surface for the scalar field*

$$f(x,y,z) = x^2 + y^2 - z.$$

So the unit vector normal to the given surface, at $(2,3,7)$*, is*

$$\frac{\nabla f(2,3,7)}{|\nabla f(2,3,7)|} = \frac{4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - \hat{\mathbf{k}}}{|4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - \hat{\mathbf{k}}|} = \frac{1}{\sqrt{53}}(4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - \hat{\mathbf{k}}).$$

$-\frac{1}{\sqrt{53}}(4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - \hat{\mathbf{k}})$ is also a correct answer. Why?

3.2 Vector Fields

Definition 3.2.1. *A* **vector field F** *is a rule associating with each point* (x,y,z) *in a region (domain) D.*

3.2.1 Examples of vector fields

1. $\mathbf{V}(x,y,z) = x^2y\hat{\mathbf{i}} - 2yz^3\hat{\mathbf{j}} + x^2z\hat{\mathbf{k}}$.
2. ∇f, where f is a scalar field.
3. The instantaneous velocity of a fluid at every point of a region in a river.

Any vector field may be written in terms of its components:

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}. \tag{3.8}$$

There are two basic concepts that arise in connection with the spatial rate of change of a vector field $\mathbf{F}$, namely the divergence of $\mathbf{F}$ and the curl of $\mathbf{F}$.

Definition 3.2.2. *The* **divergence** *of a vector field*

$$\mathbf{F} = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}}$$

is a scalar field, denoted div $\mathbf{F}$*, defined by*

$$div\ \mathbf{F} = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \cdot \left(F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}}\right). \tag{3.9}$$

We observe that in terms of the del operator ∇

$$div\mathbf{F} = \nabla \cdot \mathbf{F}.$$

Thus,

$$div\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note that $\nabla \cdot \mathbf{F} \neq \mathbf{F} \cdot \nabla$. In fact the operator $\mathbf{F} \cdot \nabla$ is meaningless alone.

Example 3.2.1. *Find the divergence of the vector field*

$$\mathbf{F} = xe^y\hat{\mathbf{i}} + e^{xy}\hat{\mathbf{j}} + \sin(yz)\hat{\mathbf{k}}.$$

Solution 3.2.1.

$$\begin{aligned} div\ \mathbf{F} = \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(e^{xy}) + \frac{\partial}{\partial z}(\sin(yz)) \\ &= e^y + xe^{xy} + y\cos(yz). \end{aligned}$$

Definition 3.2.3. *A vector field whose divergence is zero in a certain region is said to be* **solenoidal** *or* **non-divergent** *in that region.*

Roughly speaking, the divergence of a vector field is a scalar field that tells us, at each point in the region of definition, the extend to which the field diverges or explodes from that point.

Definition 3.2.4. *The* **curl** *of a vector field*

$$\mathbf{F} = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}},$$

denoted curl $\mathbf{F}$*, is the vector field defined by*

$$curl\ \mathbf{F} = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \times \left(F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}}\right) \tag{3.10}$$

Again we observe that in terms of the del operator ∇

$$curl\ \mathbf{F} = \nabla \times \mathbf{F}.$$

Hence,

$$curl\ \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{\mathbf{k}}.$$

Example 3.2.2. *Find curl* $\mathbf{F}$*, if*

$$\mathbf{F} = xyz\hat{\mathbf{i}} + x^2y^2z^2\hat{\mathbf{i}} + y^2z^3\hat{\mathbf{k}}.$$

Solution 3.2.2.

$$curl\ \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & x^2y^2z^2 & y^2z^3 \end{vmatrix} = (2yz^3 - 2x^2y^2z)\hat{\mathbf{i}} + zy\hat{\mathbf{j}} + (2xy^2z^2 - xz)\hat{\mathbf{k}}.$$

Definition 3.2.5. *A vector field with the property that its curl is identically zero is said to be* **conservative***.*

If a vector field $\mathbf{F}$ is conservative in a domain D, then there can be found a scalar field ϕ defined in D such that

$$\mathbf{F} = \nabla\phi,$$

and ϕ is called a **scalar potential function** of $\mathbf{F}$.

Exercise 3.2.1. *Show that* $\mathbf{F} = 2xy\hat{\mathbf{i}} + (x^2+1)\hat{\mathbf{j}} + 6z^2\hat{\mathbf{k}}$ *is conservative, and hence find its corresponding scalar potential* ϕ*.*

The curl of a vector field is a vector field that gives us, at each point, an indication of how the field rotates, or swirls, from that point.

3.2.2 The Laplacian

Definition 3.2.6. *The* **Laplacian** *of a scalar field* f*, denoted by* $\nabla^2 f$*, is defined by*

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{3.11}$$

The Laplacian of f is also defined as the divergence of the gradient of f:

$$\nabla^2 f = div\ (grad\ f) = \nabla\cdot(\nabla f) = \nabla^2 f.$$

If f is a scalar field, then $\nabla^2 f(x,y,z)$ is a number that tells us something about the behaviour of the scalar field in the vicinity of (x,y,z).

Definition 3.2.7. *If* ϕ *is a scalar field such that*

$$\nabla^2\phi = 0, \tag{3.12}$$

then ϕ *is said to be* **harmonic***.*

The partial differential equation (3.11) is called **Laplace equation**. It means that the average value of f in any neighborhood of (x,y,z) will be exactly equal to the value of f.

Exercise 3.2.2.
1. *Show that* $\phi = \frac{1}{r}$ *is a solution to the Laplace equation* $\nabla^2\phi = 0$*, where* $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$
2. *If* $f = f(r)$*, show that*

$$\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}.$$

Hence show that $f(r) = \alpha + \frac{\beta}{r}$ *is harmonic.*

Chapter 4

CURVILINEAR COORDINATE SYSTEMS

In many applications of vector analysis it often becomes necessary to use coordinate systems other than the Cartesian. In such situations, the representations of the gradient of a scalar field, the divergence and curl of a vector field take on completely different forms. We shall therefore discuss, in this chapter, the idea of a curvilinear coordinate system in preparation for the derivation of the corresponding expressions for the gradient, the divergence, the curl and any other related application in such a coordinate system.

Let u_1, u_2, u_3 denote the new coordinates, which can possibly be interpreted as lengths, or angles. Suppose that the new coordinates can be related to the Cartesian coordinates x, y, z by the transformation equations

$$u_1 = u_1(x, y, z), \quad u_2 = u_2(x, y, z), \quad u_3 = u_3(x, y, z), \tag{4.1}$$

such that the inverse transformation

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3) \tag{4.2}$$

exists. We call the ordered triple of numbers (u_1, u_2, u_3) the **curvilinear coordinates** of the point (x, y, z). The coordinates are generally not straight lines, as in the Cartesian coordinate system, hence the term "curvilinear".

4.1 Coordinate Surfaces and Coordinate curves

Suppose P is a point in space with curvilinear coordinates (u_1, u_2, u_3). Then the equations

$$u_1(x, y, z) = c_1, \quad u_2(x, y, z) = c_2, \quad u_3(x, y, z) = c_3 \tag{4.3}$$

define three surfaces in space each of which passes through the point P. We call the three surfaces described by (4.3) the **coordinate surfaces** intersecting at the point P and each pair of these surfaces intersect in curves called **coordinate curves**. Thus, for example, the surfaces $u_2(x, y, z) = c_2$ and $u_3(x, y, z) = c_3$ intersect in the curve on which only u_1 varies. Hence we call this curve the u_1-coordinate curve. The u_2- and u_3-coordinate curves are defined similarly. [**TO BE ILLUSTRATED IN CLASS**].

Using the inverse transformation (4.2), the position vector of a point in curvilinear coordinates now has the representation

$$\mathbf{r}(u_1, u_2, u_3) = x(u_1, u_2, u_3)\hat{\mathbf{i}} + y(u_1, u_2, u_3)\hat{\mathbf{j}} + z(u_1, u_2, u_3)\hat{\mathbf{k}} \tag{4.4}$$

It follows that the derivative $\frac{\partial \mathbf{r}}{\partial u_1}$ represents the tangent vector to the u_1-coordinate curve. Likewise, we have $\frac{\partial \mathbf{r}}{\partial u_2}$ and $\frac{\partial \mathbf{r}}{\partial u_3}$ representing the tangent vectors to the u_2 and u_3-coordinate curves, respectively. On the other hand, the normal to the surfaces The coordinate surfaces $u_i(x, y, z) = c_i$, $i = 1, 2, 3$, is the vector

$$\nabla u_i = \frac{\partial u_i}{\partial x}\hat{\mathbf{i}} + \frac{\partial u_i}{\partial y}\hat{\mathbf{j}} + \frac{\partial u_i}{\partial z}\hat{\mathbf{k}}. \tag{4.5}$$

Henceforth, we assume that the coordinates u_1, u_2, u_3 are so labeled that the vectors $\frac{\partial \mathbf{r}}{\partial u_1}, \frac{\partial \mathbf{r}}{\partial u_2}, \frac{\partial \mathbf{r}}{\partial u_3}$, in that order form a right-handed system.

Definition 4.1.1. *Whenever the vector* $\frac{d\mathbf{r}}{du_1}, \frac{d\mathbf{r}}{du_2}, \frac{d\mathbf{r}}{du_3}$ *are mutually perpendicular at every point, we say that* u_1, u_2, u_3 *comprise* **orthogonal curvilinear coordinates***.*

Any coordinate curve for u_i intersects the isotimic surface $u_i(x, y, z) = c_i$ at right angles when u_1, u_2, u_3 form orthogonal curvilinear coordinates. Consider the u_1-coordinate curve, for instance:

1. this curve is the intersection of two surfaces $u_2(x, y, z) = c_2$ and $u_3(x, y, z) = c_3$. Hence, its tangent $\frac{\partial \mathbf{r}}{\partial u_1}$ is perpendicular to both ∇u_2 and ∇u_3.
2. the vector ∇u_1 is also perpendicular to both ∇u_2 and ∇u_3, by definition of orthogonal curvilinear coordinates.

This implies that $\frac{\partial \mathbf{r}}{\partial u_1}$ is parallel to ∇u_1. Likewise $\frac{\partial \mathbf{r}}{\partial u_2}$ and $\frac{\partial \mathbf{r}}{\partial u_3}$ are parallel to ∇u_2 and ∇u_3, respectively. It follows that $\nabla u_1, \nabla u_2, \nabla u_3$ also form a right-handed system of mutually orthogonal vectors.

It is therefore natural to define a right-handed system of mutually orthogonal unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ in the u_1, u_2, u_3 coordinate curves positive directions, respectively, by:

$$\hat{\mathbf{e}}_i = \frac{\nabla u_i}{|\nabla u_i|} = \frac{\partial \mathbf{r}}{\partial u_i} \Bigg/ \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|, \qquad i = 1, 2, 3. \tag{4.6}$$

The basic difference between curvilinear coordinates and Cartesian coordinates is that in the Cartesian coordinates, the unit vectors (in the respective directions of the coordinate curves) $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are constants for all points of space and are equal to $\hat{\mathbf{1}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, respectively; in any other coordinate system, the unit vectors will, generally speaking, change with position in space.

So at each point in space, we can define a set of unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ such that any arbitrary vector $\mathbf{F}$ located at a particular point in space may be written in curvilinear component form as

$$\mathbf{F} = F_1 \hat{\mathbf{e}}_1 + F_2 \hat{\mathbf{e}}_2 + F_3 \hat{\mathbf{e}}_3. \tag{4.7}$$

In that case the coordinate curves are mutually orthogonal at every point and so are the corresponding unit vectors. This is expressed by the conditions

$$\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0, \quad \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3 = 0, \quad \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 = 0. \tag{4.8}$$

We also deduce that if u_1, u_2, u_3 form a right-handed system then the corresponding unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ also form a right-handed system and this is expressed by the conditions

$$\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3, \quad \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_2, \quad \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1. \tag{4.9}$$

4.1.1 Scale factors

Vector operations in general orthogonal curvilinear coordinates are usually expressed in terms of functions known as scale factors, denoted by h_i, $i = 1, 2, 3$.

Definition 4.1.2. *The* **scale factor** h_i *is defined to be the rate at which arc length increases on the* i^{th} *coordinate curve, with respect to* u_i. *That is,*

$$h_i = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|, \quad i = 1, 2, 3. \tag{4.10}$$

Equations (4.6) and (4.10) lead us to

$$\frac{\partial \mathbf{r}}{\partial u_i} = h_i \hat{\mathbf{e}}_i. \tag{4.11}$$

We can deduce another formula for the scale factors h_i by making the following observations:

1. $|\nabla u_i|$ is the rate of change of u_i with respect to distance in the direction of ∇u_i.
2. The direction of ∇u_i is the direction of the u_i-coordinate curve.

It follows that

$$|\nabla u_i| = \frac{\partial u_i}{\partial s_i} = \frac{1}{h_i}.$$

Hence

$$h_1 = \frac{1}{|\nabla u_1|}, \quad h_2 = \frac{1}{|\nabla u_2|}, \quad h_3 = \frac{1}{|\nabla u_3|}. \tag{4.12}$$

Example 4.1.1. *Consider the curvilinear coordinate system (u_1, u_2, u_2) defined for $z \geq 0$ by*

$$x = u_1 - u_2, \quad y = u_1 + u_2, \quad z = u_3^2.$$

1. *Compute the corresponding unit vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and hence verify that the system is orthogonal and right-handed.*
2. *Compute the corresponding scale factors.*

Solution 4.1.1. *1.*

$$\hat{\mathbf{e}}_1 = \frac{\partial \mathbf{r}}{\partial u_1} \Bigg/ \left|\frac{\partial \mathbf{r}}{\partial u_1}\right| = \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}, \quad \hat{\mathbf{e}}_2 = \frac{\partial \mathbf{r}}{\partial u_2} \Bigg/ \left|\frac{\partial \mathbf{r}}{\partial u_2}\right| = \frac{-\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}, \quad \hat{\mathbf{e}}_3 = \frac{\partial \mathbf{r}}{\partial u_3} \Bigg/ \left|\frac{\partial \mathbf{r}}{\partial u_3}\right| = \frac{2u_3\hat{\mathbf{k}}}{|2u_3|} = \hat{\mathbf{k}}.$$

Now

$$\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0, \quad \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3 = 0, \quad \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 = 0,$$

hence the system is orthogonal. Further,

$$\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3, \quad \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2, \quad \hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1,$$

and so the system is right-handed.

2. *The scale factors are*

$$h_1 = \frac{1}{|\nabla u_1|} = \sqrt{2}, \quad h_2 = \frac{1}{|\nabla u_2|} = \sqrt{2}, \quad h_3 = \frac{1}{|\nabla u_3|} = 2u_3.$$

Exercise 4.1.1. 1. *Consider a spherical coordinate system* (r, ϕ, θ) *defined by the transformation equations*

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi,$$

where

$$0 \le r \le \infty, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

Determine the scale factors and express the unit vectors $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_\theta$ *in terms of* $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. *Hence show that a spherical coordinate system is orthogonal and right-handed.*

2. *Consider a cylindrical coordinate system* (r, θ, z) *defined by the transformation equations*

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

where

$$0 \le r \le \infty, \quad 0 \le \theta \le 2\pi.$$

(a) *Determine the scale factors and express the unit vectors* $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z$ *in terms of* $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. *Hence show that a cylindrical coordinate system is orthogonal and right-handed.*

(b) *Represent the vector* $\mathbf{F} = z\hat{\mathbf{i}} - 2x\hat{\mathbf{j}} + y\hat{\mathbf{k}}$ *in cylindrical coordinates.*

The scale factors allow us to write the general formulas for arc length, area, volume, gradient, divergence, curl, Laplacian, etc. in terms of curvilinear coordinates.

4.1.2 Arc length in general orthogonal curvilinear coordinates

In the general orthogonal curvilinear coordinates

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3. \tag{4.13}$$

Hence the differential element of arc length can be expressed as

$$ds = |d\mathbf{r}| = \left|\frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3\right|. \tag{4.14}$$

Combining equations (4.7) and (4.9) we deduce that the displacement vector $d\mathbf{r}$ can be expressed in terms of the scale factors by

$$d\mathbf{r} = h_1 du_1 \hat{\mathbf{e}}_1 + h_2 du_2 \hat{\mathbf{e}}_2 + h_3 du_3 \hat{\mathbf{e}}_3. \tag{4.15}$$

Exercise 4.1.2. *Verify (4.15).*

Clearly, if the curve C is along the u_1-coordinate curve, then

$$ds = ds_1 = h_1 du_1. \tag{4.16}$$

We can similarly define the differential elements of length along the u_2- and u_1-coordinate curves as

$$ds_2 = h_2 du_2, \tag{4.17}$$

$$ds_3 = h_3 du_3, \tag{4.18}$$

respectively.

Example 4.1.2. *1. In the Cartesian coordinate system the differential elements of arc length along the coordinate curves are given by*

$$ds_1 = h_x dx = dx, \quad ds_2 = h_y dy = dy, \quad ds_3 = h_z dz = dz.$$

2. In the spherical coordinate system the differential elements of arc length along the coordinate curves are given by

$$ds_1 = h_r dr = dr, \quad ds_2 = h_\phi d\phi = r d\phi, \quad ds_3 = h_\theta d\theta = r \sin\phi d\theta.$$

Exercise 4.1.3. *1. Obtain the differential elements of arc length along the coordinate curves in the cylindrical coordinate system.*

2. By first finding the square of the element of arc length, $(ds)^2$, in spherical coordinates determine the corresponding scale factors, h_r, h_ϕ, h_θ.

4.1.3 The gradient in curvilinear coordinates

The component of the gradient of a scalar field $f(u_1, u_2, u_3)$ in the direction of the unit vector $\hat{\mathbf{e}}_i$, denoted *grad* $f = \nabla f$, is given by $\frac{df}{ds_1}$ (the rate of change of f with respect to distance in the $\hat{\mathbf{e}}_i$ direction). Since $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ are mutually orthogonal unit vectors, we can readily express *grad* f as

$$\nabla f = \frac{df}{ds_1}\hat{\mathbf{e}}_1 + \frac{df}{ds_2}\hat{\mathbf{e}}_2 + \frac{df}{ds_3}\hat{\mathbf{e}}_3 = \frac{du_1}{ds_1}\frac{\partial f}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{du_2}{ds_2}\frac{\partial f}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{du_3}{ds_3}\frac{\partial f}{\partial u_3}\hat{\mathbf{e}}_3. \tag{4.19}$$

Introducing the scale factors we have

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{\mathbf{e}}_3. \tag{4.20}$$

This indicates the operator equivalence

$$\nabla = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3} \tag{4.21}$$

in the coordinate system (u_1, u_2, u_3).

Example 4.1.3. *Compute the gradient of the scalar field*

$$f(u_1, u_2, u_3) = u_1 u_2 + u_3^2$$

in the coordinate system (u_1, u_2, u_3) *defined by*

$$x = u_1 - u_2, \qquad y = u_1 + u_2, \qquad z = u_3^2.$$

Solution 4.1.2.

$$\nabla f = \left(\frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}\right)\left(u_1 u_2 + u_3^2\right) = \frac{1}{\sqrt{2}}u_2\hat{\mathbf{e}}_1 + \frac{1}{\sqrt{2}}u_1\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3.$$

Exercise 4.1.4. *Derive the expressions for div* $\mathbf{F}$, *curl* $\mathbf{F}$ *and* $\nabla^2 F$ *in generalised curvilinear coordinates in terms of the scale factors and the unit vectors.*

Chapter 5

INTEGRAL CALCULUS OF SCALAR AND VECTOR FIELDS

5.1 Line Integrals of Scalar Fields

Let f be a scalar field defined in a domain D of the xyz-space and let

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}, \qquad t_1 \le t \le t_2, \tag{5.1}$$

represent a space curve C that lies in D. We assume that C is a smooth curve and that f is continuous on C.

Recall that the arc length of C is given by

$$s(t) = \int_{t_1}^{t} |\mathbf{r}'(\tau)| d\tau, \qquad (t_1 \le t \le t_2) \tag{5.2}$$

so that $ds = |\mathbf{r}'(t)|dt$.

Definition 5.1.1. *The* **line integral** *of f on C with respect to arc length, denoted by $\int_C f ds$, is the integral*

$$\int_C f(x,y,z)ds = \int_{t_1}^{2} f[x(t), y(t), z(t)]|\mathbf{r}'(t)|dt \tag{5.3}$$

where

$$|\mathbf{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.$$

Example 5.1.1. *Evaluate the line integral of*

$$f = x + 2y$$

on a straight line

$$y = 2x$$

from the origin to the point $(1, 2)$.

Solution 5.1.1. C *is the line, represented by the vector equation*

$$\mathbf{r}(t) = t\hat{\mathbf{i}} + 2t\hat{\mathbf{j}}, \quad 0 \le t \le 1.$$

Thus,

$$\int_C (x + 2y)ds = \int_0^1 (t + 4t)|\mathbf{r}'(t)|dt = \int_0^1 (t + 4t)\sqrt{5}dt = \frac{5\sqrt{5}}{2}.$$

Example 5.1.2. *Evaluate the line integral*

$$\int_C xyz\ ds,$$

where C *is the helix*

$$\mathbf{r}(t) = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + t\hat{\mathbf{k}}, \quad 0 \le t \le 2\pi.$$

Solution 5.1.2. *Here*

$$\mathbf{r}'(t) = -\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

so that $|\mathbf{r}'(t)| = \sqrt{2}$. *Thus*

$$\int_C xyz\ ds = \sqrt{2}\int_0^{2\pi} t\cos t\sin t dt$$

$$= \sqrt{2}\left[\frac{t}{2}\sin^2 t\right]_0^{2\pi} - \frac{\sqrt{2}}{2}\int_0^{2\pi} \sin^2 t dt$$

$$= -\frac{\sqrt{2}}{2}\pi.$$

Exercise 5.1.1. *In each of the following problems, calculate the line integral along the given curve:*

1. $\int_C (x - y)^2 dx$ *along the parabola* $y = x - x^2/4$ *from* $(0, 0)$ *to* $(4, 0)$.
2. $\int_C (x - y)^2 dx$ *around a circle of radius* a *traced counterclockwise.*
3. $\int_C (x^2 + y^2)dy$ *around the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *traced counterclockwise.*
4. $\int_C (xz + yz + xy)dy$ *around the intersection of the cylinder* $x^2 + y^2 = 1$ *and the plane* $y + z = 1$ *in the clockwise direction as viewed from the origin.*

5.2 Line Integrals of Vector Fields

Definition 5.2.1. *Let*

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}} \tag{5.4}$$

be a vector field defined and continuous in a domain D, and let C be a smooth curve in D represented by

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}, \qquad t_1 \le t \le t_2.$$

The line integral of $\mathbf{F}$ on C, denoted $\int_C \mathbf{F}\cdot d\mathbf{r}$, is the integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}[x(t),y(t),z(t)]\cdot\frac{d\mathbf{r}}{dt}dt. \tag{5.5}$$

Observe that the integral (5.5) is taken along the positive direction on the curve C. If the curve is traversed in the opposite direction, the integral changes sign. If

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}},$$

the integral (5.5) can also be written as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C F_1dx + F_2dy + F_3dz. \tag{5.6}$$

Example 5.2.1. *Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where*

$$\mathbf{F} = x^2\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (xz-y)\hat{\mathbf{k}},$$

from $(0,0,0)$ to $(1,2,4)$

1. *along the line segment joining the two points.*
2. *along the curve given parametrically by*

$$x = t^2, \qquad y = 2t, \qquad z = 4t^3.$$

Solution 5.2.1. *1. The parametric equations of the straight line joining $(0,0,0)$ to $(1,2,4)$ are*

$$x = t \qquad y = 2t, \qquad z = 4t.$$

Hence,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C x^2dx + ydy + (xz-y)dz \\ &= \int_0^1 t^2dt + (2t)(2dt) + (4t^2-2t)(4dt) \\ &= \int_0^1 (17t^2-4t)dt \\ &= \frac{11}{3}.\end{aligned}$$

2. **Exercise**.

Observe that the line integral has been defined without reference to the parametrization of the curve, so its value will depend only on the field $\mathbf{F}$ and the oriented curve C, and not on the choice of the parameter.

Example 5.2.2. *Calculate the line integral of*

$$\mathbf{F}(x,y,z) = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

along the helix

$$C: \quad \mathbf{r}(\theta) = \cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} + \theta\hat{\mathbf{k}}, \qquad 0 \le \theta \le \pi.$$

Solution 5.2.2.

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \\ &= \int_0^\pi (-1+t)dt \\ &= -\pi + \frac{\pi^2}{2}.\end{aligned}$$

If $\mathbf{F}$ represents a force field, then the work done by $\mathbf{F}$ in moving a particle from an initial point to a final point of an oriented curve C is given by

$$W = \int_C \mathbf{F}\cdot d\mathbf{r}. \tag{5.7}$$

Example 5.2.3. *Calculate the work done by a force*

$$\mathbf{F}(x,y) = x^2y\hat{\mathbf{i}} + (x^2+y)\hat{\mathbf{j}}$$

in moving a particle from the origin to the point $(2,4)$ *along the parabola* $y = x^2$*.*

Solution 5.2.3. *We can write the vector equation of the parabola as*

$$\mathbf{r}(x) = x\hat{\mathbf{i}} + x^2\hat{\mathbf{j}},$$

and along the parabola the force field $\mathbf{F}$ *is*

$$\mathbf{F} = x^4\hat{\mathbf{i}} + 2x^2\hat{\mathbf{j}}.$$

Hence

$$\begin{aligned}\text{Work} \quad \text{done} &= \int_C \mathbf{F}\cdot d\mathbf{r} \\ &= \int_C \mathbf{F}\cdot \frac{d\mathbf{r}}{dx}dx \\ &= \int_0^2 (4x^4\hat{\mathbf{i}} + 2x^2\hat{\mathbf{j}})\cdot(\hat{\mathbf{i}} + 2x\hat{\mathbf{j}})dx \\ &= \int_0^2 (x^4 + 4x^3)dx = \frac{112}{5} \quad \textit{units}.\end{aligned}$$

Exercise 5.2.1. *Find the work done by the force field*

$$\mathbf{F} = x^2\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}$$

in moving a particle along the curve

$$x = t^2, \quad y = t^2 + 1, \quad z = t^3, \quad \text{from} \quad t = 0 \quad \text{to} \quad t = 2.$$

Definition 5.2.2. *If the curve* C *is closed, that is, its initial and final points coincide, then the line integral of a vector field* $\mathbf{F}$ *around* C*, denoted* $\oint_C \mathbf{F}\cdot d\mathbf{r}$*, is called the* **circulation** *of* $\mathbf{F}$ *about* C*.*

Example 5.2.4. *Evaluate the line integral of*

$$\mathbf{F} = x^2\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

around the circle $C: \quad x^2 + y^2 = 4.$

Solution 5.2.4. *We represent the circle* C *in polar coordinates as*

$$x = 2\cos\theta, \qquad y = 2\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

Then we have

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} [(4\cos^2\theta)(-2\sin\theta)d\theta + 2\sin\theta(2\cos\theta)d\theta] \\ &= \int_0^{2\pi} (-8\cos^2\theta\sin\theta + 4\sin\theta\cos\theta)d\theta = 0.\end{aligned}$$

Exercise 5.2.2. *Evaluate the integral of*

$$\mathbf{F} = (x + y^2)\hat{\mathbf{i}} + (x + z)\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

from the origin to the point $(1, -1, 1)$

1. *along the straight line joining the points*
2. *along the curve* $\mathbf{r}(t) = t\hat{\mathbf{i}} - t^2\hat{\mathbf{j}} + t^3\hat{\mathbf{k}}$, $(0 \leq t \leq 1)$.

The exercise above shows that the value of the line integral from $(0, 0, 0,)$ to $(1, -1, 1)$ depends on the path of integration.

5.2.1 Properties of line integrals

Suppose f and g are scalar fields and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, all defined and continuous in a domain containing the smooth curve $C: \quad \mathbf{r} = \mathbf{r}(t) \quad (t_1 \leq t \leq t_2)$. Then for any constants c_1 and c_2, we have

1. **The linearity property of line integrals**:

$$\int_C (c_1 f + c_2 g) ds = c_1 \int_C f ds + c_2 \int_C g ds \tag{5.8}$$

   and

$$\int_C (c_1\mathbf{F} + c_2\mathbf{G}) \cdot d\mathbf{r} = c_1 \int_C \mathbf{F} \cdot d\mathbf{r} + c_2 \int_C \mathbf{G} \cdot d\mathbf{r}. \tag{5.9}$$

2. **The additive property of line integrals**. Suppose the curve C is comprised of n connected smooth curves

   $C_1 : \mathbf{r} = \mathbf{r}_1(t) \quad (t_1 \leq t \leq t_2), \quad C_2 : \mathbf{r} = \mathbf{r}_2(t) \quad (t_2 \leq t \leq t_3), \cdots, C_n : \mathbf{r} = \mathbf{r}_n(t) \quad (t_n \leq t \leq t_{n+1})$.

   Then

$$\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds + \cdots + \int_{C_n} f ds \tag{5.10}$$

   and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot d\mathbf{r}. \tag{5.11}$$

Example 5.2.5. *Evaluate the integral*

$$\int_C xdx - zdy + 2ydz,$$

where C *consists of the line segments from the origin to the point* $(1,1,0)$ *and from* $(1,1,0)$ *to* $(1,1,2)$.

Solution 5.2.5. *Let* C_1 *be the line segment from* $(0,0,0)$ *to* $(1,1,0)$ *and* C_2 *be the line segment from* $(1,1,0)$ *to* $(1,1,2)$. *Then on* C_1, $x = y$, $z = 0$, *so that* $dx = dy$, $dz = 0$. *On* C_2, $x = 1 = y$ *so that* $dx = 0 = dy$. *Hence we have*

$$\int_C xdx - zdy + 2ydz = \int_{C_1} xdx + \int_{C_2} 2ydz$$
$$= \int_0^1 xdx + \int_0^2 2dz = \frac{9}{2}.$$

Example 5.2.6. *Evaluate the integral*

$$I = \oint_C (x^2 + y)dx + (y^2 + z)dy + (z^2 + x)dz$$

around the closed curve c consisting of the line segments

$$C_1: \quad x + z = 1 \quad (0 \le x \le 1, \quad y = 0) \qquad \text{and} \qquad C_2: \quad x + y = 1 \quad (0 \le y \le 1, \quad z = 0),$$

and the quarter circle

$$C_3: \quad y^2 + z^2 = 1 \quad (y \ge 0, z \ge 0)$$

in the counterclockwise direction.

Solution 5.2.6. *On* C_1 *we have* $y = 0$, $z = 1 - x$ *so that* $dy = 0$ *and* $dz = -dx$. *Hence, using* x *as a parameter, we find*

$$I_1 = \int_{C_1} (x^2 + y)dx + (y^2 + z)dy = \int_0^1 x^2 dx + [(1-x)^2 + x](-dx)$$
$$= \int_0^1 (x - 1)dx = -\frac{1}{2}.$$

On C_2, *using* y *as a parameter, we have* $x = 1 - y$, $z = 0$, *so that* $dx = -dy$, $dz = 0$. *Hence*

$$I_2 = \int_{C_2} (x^2 + y)dx + (y^2 + z)dy = \int_0^1 [(1-y)^2 + y](-dy) + y^2 dy$$
$$= \int_0^1 (y - 1)dy = -\frac{1}{2}.$$

Finally, on C_3 where $x = 0$, we set $y = \cos\theta$, $z = \sin\theta$ $(0 \le \theta \le \pi/2)$. Then

$$I_3 = \int_{C_3} (y^2 + z)dy + z^2 dz$$

$$= \int_0^{\pi/2} [(\cos^2\theta + \sin\theta)(-\sin\theta) + \sin^2\theta\cos\theta]d\theta$$

$$= \int_0^{\pi/2} (-\cos^2\theta\sin\theta - \sin^2\theta + \sin\theta^2\cos\theta)d\theta$$

$$= \left[\frac{\cos^3\theta}{3} - \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + \frac{\sin^3\theta}{3}\right]_0^{\pi/2} = -\frac{\pi}{4}.$$

Thus the line integral around the closed curve C is

$$I_1 + I_2 + I_3 = -1 - \frac{\pi}{4}.$$

Exercise 5.2.3. *Evaluate the line integral $\int_C ydx - xdy + zdz$ around the curve of intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $z - y = a$ taken in the counterclockwise direction.*

5.2.2 Line integrals independent of path

Theorem 5.2.1. *Let $\mathbf{F}$ be continuous in a domain D. The line integral of $\mathbf{F}$ is independent of the path in D if and only if*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \tag{5.12}$$

for any piecewise smooth simple closed curve C in D.

5.3 Oriented Surfaces

Just as it is possible to write the equation of a space curve C in parametric form, giving x, y and z as functions of a *single* parameter t, it is possible to represent a two-dimensional surface S parametrically by giving x, y and z as functions of two parameters, say u and v. Parametrically, then, a surface S is represented by equations of the form

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v) \tag{5.13}$$

where u and v are two parameters which range over some domain D in the uv-plane. The parametric equations (5.13) can be combined into a single vector equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}. \tag{5.14}$$

As the parameters u and v vary over D, the tip of the position vector $\mathbf{r}(u,v)$ generates the surface S.

Example 5.3.1. *A sphere of radius a and center at the origin may be represented parametrically by the equations*

$$x = a\sin\phi\cos\theta, \qquad y = a\sin\phi\sin\theta, \qquad z = a\cos\phi \tag{5.15}$$

where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

5.3.1 Normal vector on a surface

Let S be a surface represented parametrically by

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(u,v)$$

where the functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ are continuous and have continuous derivatives in the domain D of the uv-plane. Let us consider the position vector $\mathbf{r}(u,v)$ of a point (u_0,v_0) on S. Then

$$\mathbf{r}(u,v_0) = x(u,v_0)\mathbf{i} + y(u,v_0)\mathbf{j} + z(u,v_0)\mathbf{k}$$

represents a curve on the surface with u as the parameter, whose tangent vector is given by

$$\frac{\partial \mathbf{r}(u,v_0)}{\partial u}.$$

Similarly,

$$\mathbf{r}(u_0,v) = x(u_0,v)\mathbf{i} + y(u_0,v)\mathbf{j} + z(u_0,v)\mathbf{k}$$

represents another curve on the surface with its tangent vector given by

$$\frac{\partial \mathbf{r}(u_0,v)}{\partial u}.$$

Since both tangent vectors are tangent to curves in the surface S, they are tangent to the surface itself at the point (u_0,v_0). It follows that the vector

$$\frac{\partial \mathbf{r}(u_0,v_0)}{\partial u} \times \frac{\partial \mathbf{r}(u_0,v_0)}{\partial v}$$

is normal to the surface S at (u_0, v_0). In fact, a unit vector normal the surface S at (u_0, v_0) is

$$\hat{\mathbf{n}} = \frac{\frac{\partial \mathbf{r}(u_0,v_0)}{\partial u} \times \frac{\partial \mathbf{r}(u_0,v_0)}{\partial v}}{|\frac{\partial \mathbf{r}(u_0,v_0)}{\partial u} \times \frac{\partial \mathbf{r}(u_0,v_0)}{\partial v}|}. \tag{5.16}$$

We have, so far, derived two ways of computing the unit vector normal to the surface:

1. If the surface S is specified non-parametrically by $f(x, y, z) = c$, the

   $$\frac{grad\ f}{|grad\ f|} = \frac{\nabla f}{|\nabla f|}$$

   is a unit vector normal to S.

2. If the surface S is given parametrically through equations (5.13) or (5.14), then (5.16) is a unit vector normal to S.

Example 5.3.2. *Find a unit vector normal to the surface represented the equation*

$$\mathbf{r}(u, v) = u(\cos v\hat{\mathbf{i}} + \sin v\hat{\mathbf{j}}) + (1 - u^2)\hat{\mathbf{k}}, \quad u \geq 0, \quad 0 \leq v \leq 2\pi.$$

Solution 5.3.1. *The unit normal vector is*

$$\hat{\mathbf{n}} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}|}$$

$$= \frac{(\cos v\hat{\mathbf{i}} + \sin v\hat{\mathbf{j}} - 2u\hat{\mathbf{k}}) \times u(-\sin v\hat{\mathbf{i}} + \cos v\hat{\mathbf{j}})}{|(\cos v\hat{\mathbf{i}} + \sin v\hat{\mathbf{j}} - 2u\hat{\mathbf{k}}) \times u(-\sin v\hat{\mathbf{i}} + \cos v\hat{\mathbf{j}})|} = \frac{2u^2(\cos v\hat{\mathbf{i}} + \sin V\hat{\mathbf{j}}) + u\hat{\mathbf{k}}}{u\sqrt{4u^2 - 1}}$$

$$= \frac{2u(\cos v\hat{\mathbf{i}} + \sin V\hat{\mathbf{j}}) + u\hat{\mathbf{k}}}{\sqrt{4u^2 - 1}}.$$

We say that a surface S, defined by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

is **smooth** if $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are continuous and the normal vector $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is not zero at any point on S. If a surface is not smooth but consists of a finite number of surfaces each of which is smooth, then the surface is said to be **piecewise smooth**.

Example 5.3.3. *The surface defined by a sphere is smooth and the faces of a cube constitute a piecewise smooth surface.*

A surface S is said to be **closed** if it has no boundary, otherwise it is a **two-sided** surface. At every point on a smooth surface there will, of course, be two choices for the unit normal vector $\hat{\mathbf{n}}$:

$$\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} \quad \text{and} \quad -\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} \quad \text{or} \quad \frac{\frac{\partial \mathbf{r}(u_0,v_0)}{\partial u} \times \frac{\partial \mathbf{r}(u_0,v_0)}{\partial v}}{|\frac{\partial \mathbf{r}(u_0,v_0)}{\partial u} \times \frac{\partial \mathbf{r}(u_0,v_0)}{\partial v}|} \quad \text{and} \quad -\frac{\frac{\partial \mathbf{r}(u_0,v_0)}{\partial u} \times \frac{\partial \mathbf{r}(u_0,v_0)}{\partial v}}{|\frac{\partial \mathbf{r}(u_0,v_0)}{\partial u} \times \frac{\partial \mathbf{r}(u_0,v_0)}{\partial v}|}.$$

If S is *two-sided*, that is, it has a boundary, then (following the right-hand rule) the side of S on the "upward" direction is called "positive" $\hat{\mathbf{n}}$ side of the surface. Otherwise, if S is closed then by convention $\hat{\mathbf{n}}$ is chosen to point outward.

Exercise 5.3.1. *1. Find a parametric representation of each of the following surfaces:*

(a) The pale $ax + by + cz + d = 0$.

(b) The parabolic cylinder $x = x^2$.

(c) The elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - z$.

(d) The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$.

2. Determine a unit normal vector to each of the following surfaces, at the given point:

(a) $\mathbf{r}(u,v) = u\cos v\hat{\mathbf{i}} + u\sin v\hat{\mathbf{j}} + u^2\hat{\mathbf{k}}$ at $(0,-1,2)$.

(b) $\mathbf{r}(u,v) = u\cos v\hat{\mathbf{i}} + u\sin v\hat{\mathbf{j}} + u\hat{\mathbf{k}}$ at $(\sqrt{3},1,2)$.

(c) $\mathbf{r}(\phi,\theta) = \sqrt{2}(\sin\phi\cos\theta)\hat{\mathbf{i}} + 2\sqrt{2}(\sin\phi\sin\theta)\hat{\mathbf{j}} + (\sqrt{3}\cos\phi)\hat{\mathbf{k}}$ at $(1/2,1,3/2)$.

(d) $\mathbf{r}(u,v) = 2(u+v)\hat{\mathbf{i}} + (u-v)\hat{\mathbf{j}} + uv\hat{\mathbf{k}}$ at $(2,-3,-2)$.

(e) $\mathbf{r}(u,v) = 2(\sin u\cosh v)\hat{\mathbf{i}} + 3(\cos u\cosh v)\hat{\mathbf{j}} + (\sinh v)\hat{\mathbf{k}}$ at $(1,3\sqrt{3}/2,0)$.

5.3.2 Surface area

Definition 5.3.1. *Let S be a smooth surface represented by*

$$\mathbf{r} = \mathbf{r}(u,v),$$

where $\mathbf{r}(u,v)$ is continuously differentiable in D. The area of the surface S is given by

$$A = \int\int_D \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| dudv. \tag{5.17}$$

The formula

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| dudv \tag{5.18}$$

is called the element of surface area on S.

Example 5.3.4. *Find the surface area of the surface defined by the equations*

$$x = \cos\theta, \qquad y = \sin\theta, \qquad z = t$$

for $0 \le \theta \le 2\pi$, $0 \le t \le 1$.

Solution 5.3.2. *The required surface area is*

$$\begin{aligned} A &= \int\int_D \left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial t}\right| d\theta dt \\ &= \int_0^1 \int_0^{2\pi} |(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) \times (\hat{\mathbf{k}})| d\theta dt \\ &= \int_0^1 \int_0^{2\pi} |(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})| d\theta dt \\ &= \int_0^1 \int_0^{2\pi} (\cos^2\theta + \sin^2\theta)^{1/2} d\theta dt = 2\pi. \end{aligned}$$

Exercise 5.3.2.

1. *Verify that the surface area of a sphere of radius a parametrized in terms of its latitude and longitude angles* ϕ *and* θ *through the equations:*

$$x = a\sin\phi\cos\theta, \qquad y = a\sin\phi\sin\theta, \qquad z = a\cos\phi$$

where $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$, *is* $4\pi a^2$.

2. *Find the surface area of a circular cone*

$$\mathbf{r}(t,\theta) = t\cos\theta\hat{\mathbf{i}} + t\sin\theta\hat{\mathbf{j}} + t\hat{\mathbf{k}}, \qquad 0 \le t \le a, \quad 0 \le \theta \le 2\pi.$$

If the surface is represented by the equation of the form

$$z = z(x,y)$$

where (x,y) ranges over a domain D^* on the x,y-plane, we may use x and y as parameters and write

$$\mathbf{r}(x,y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z(x,y)\hat{\mathbf{k}}.$$

then

$$\frac{\partial \mathbf{r}}{\partial x} = \hat{\mathbf{i}} + \frac{\partial z}{\partial x}\hat{\mathbf{k}}, \qquad \frac{\partial \mathbf{r}}{\partial y} = \hat{\mathbf{j}} + \frac{\partial z}{\partial y}\hat{\mathbf{k}}$$

so that

$$\begin{aligned} \left|\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right| &= \left| -\frac{\partial z}{\partial x}\hat{\mathbf{i}} - \frac{\partial z}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}\right| \\ &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}. \end{aligned}$$

Hence, alternatively, the surface area of the surface S is given by

$$A = \int\int_{D^*} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy \tag{5.19}$$

so that

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy. \tag{5.20}$$

Notice that the integral (5.19) is to be integrated over the domain D^* which is the projection of S onto the x, y-plane. The formula (5.20) has an interesting an important application. Recall that a unit normal vector on the surface is given by

$$\hat{\mathbf{n}} = \frac{-\frac{\partial z}{\partial x}\hat{\mathbf{i}} - \frac{\partial z}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^{2+1}}}.$$

If γ denotes the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{k}}$, then

$$\cos\gamma = \hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}.$$

Hence, by (5.21) we have

$$dS = \frac{dxdy}{\cos\gamma} \tag{5.21}$$

and so (5.19) can be written as

$$A = \int\int_{D^*} \frac{dxdy}{\cos\gamma}. \tag{5.22}$$

5.4 Surface Integrals

We now study the integration of a scalar or a vector field on a surface.

5.4.1 Surface integral of a scalar field

Definition 5.4.1. *Let f be a scalar field defined and continuous in a domain D. Let S be a smooth surface in D represented by*

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k},$$

where $\mathbf{r}(u,v)$ is continuously differentiable over S. The **surface integral** *of f on S is defined by*

$$\int\int_S f(x,y,z)dS = \int\int_S f[x(u,v), y(u,v), z(u,v)]\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| dudv. \tag{5.23}$$

If S is piecewise smooth, we define the surface integral of f on S as the sum of the integrals over the pieces of smooth surfaces comprising S.

for a surface that is represented by an equation of the form $z = z(x,y)$, (x, vy) ranges over the projection D^* of the surface on the x,y-plane, the integral (5.23) can be written as

$$\int\int_S f(x,y,z)dS = \int\int_S f(x,y,z(x,y))\frac{dxdy}{\cos\gamma} \tag{5.24}$$

where

$$\cos\gamma = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}.$$

Similarly, if the surface can be represented by the equation of the form $x = x(y,z)$ or $y = y(x,z)$, the integral (5.23) can be written as

$$\int\int_S f(x,y,z)dS = \int\int_S f(x(y,z),y,z)\frac{dydz}{\cos\alpha} \tag{5.25}$$

or

$$\int\int_S f(x,y,z)dS = \int\int_S f(x,y(x,z),z)\frac{dxdz}{\cos\beta} \tag{5.26}$$

where

$$\cos\alpha = \frac{1}{\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}}, \qquad \cos\beta = \frac{1}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}}.$$

Example 5.4.1. *Find the surface integral of*

$$f(x,y,z) = xy + z$$

on the upper half of a sphere of radius a.

Solution 5.4.1. *We represent the surface by*

$$S: \quad \mathbf{r}(\phi,\theta) = a\sin\phi\cos\theta\hat{\mathbf{i}} + a\sin\phi\sin\theta\hat{\mathbf{j}} + a\cos\phi\hat{\mathbf{k}},$$

where

$$0 \le \phi \le \pi/2, \qquad 0 \le \theta \le 2\pi.$$

Then

$$\begin{aligned}\int\int_S f dS &= \int\int_S [(a\sin\phi\cos\theta)(a\sin\phi\sin\theta) + a\cos\phi]\left|\frac{\partial \mathbf{r}}{\partial \phi}\times\frac{\partial \mathbf{r}}{\partial \theta}\right| d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} [(a\sin\phi\cos\theta)(a\sin\phi\sin\theta) + a\cos\phi]a^2\sin\phi d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} (a^4\sin^3\phi\cos\theta\sin\theta + a^3\cos\phi\sin\phi)d\phi d\theta = \pi a^3.\end{aligned}$$

Surface integrals of scalar fields occur in many physical problems. For example, suppose we have a thin sheet of material in the shape of a surface S whose density at each point (x,y,z) is given by $\rho(x,y,z)$. Then the mass of the material is given by the surface integral

$$M = \int\int_S \rho(x,y,z)dS. \tag{5.27}$$

5.4.2 Surface integral of a vector field

Definition 5.4.2. *Let $\mathbf{F}$ be a vector field defined and continuous in a domain D, and let S be a smooth oriented surface in D represented by $\mathbf{r} = \mathbf{r}(u,v)$ where $\mathbf{r}(u,v)$ is continuously differentiable in D. The* **surface integral** *of $\mathbf{F}$, also known as the* **flux** *of $\mathbf{F}$, on S is given by*

$$\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \int\int_S \mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}dudv. \tag{5.28}$$

If S is piecewise smooth, we define the surface integral (5.28) as the sum of the integrals over the smooth surfaces comprising the surface S. When S is closed it is customary to take $\hat{\mathbf{n}}$ to be the outward unit normal vector.

If the surface S is represented by $z = z(x, y)$, $(x, y) \in D^*$, so that $dS = \frac{dxdy}{\cos\gamma}$, then the integral (5.23) can also be written in the form

$$\int\int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \int\int_S \mathbf{F} \cdot \hat{\mathbf{n}} \frac{dxdy}{\cos\gamma}. \tag{5.29}$$

Corresponding formulas can also be obtained using the relation

$$dS = \frac{dydz}{\cos\alpha} \quad \text{or} \quad dS = \frac{dxdz}{\cos\beta},$$

whenever the surface S can be represented by $x = x(y, z)$ or $y = y(x, z)$.

Example 5.4.2. *Calculate the flux of the vector field*

$$\mathbf{F}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 3z\hat{\mathbf{k}}$$

on the surface

$$S: \quad \mathbf{r}(u, v) = a(\cos u\hat{\mathbf{i}} + \sin u\hat{\mathbf{j}}) + v\hat{\mathbf{k}}, \qquad 0 \le u \le \pi, \qquad 0 \le v \le \pi.$$

Solution 5.4.2. *On the surface S we have*

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = a(\cos u\hat{\mathbf{i}} + \sin u\hat{\mathbf{j}})$$

and

$$\mathbf{F} = a(\cos u\hat{\mathbf{i}} + \sin u\hat{\mathbf{j}}) + 3v\hat{\mathbf{k}}.$$

Hence

$$Flux \quad of \quad \mathbf{F} = \int\int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \int_0^\pi \int_0^\pi \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} dudv = a^2\pi^2.$$

Exercise 5.4.1.

1. *Given*

$$\mathbf{F} = x\hat{\mathbf{i}} - y\hat{\mathbf{j}},$$

find the value of $\int\int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$ over the closed surface S made up of the planes $z = 0$, $z = 1$ and the cylinder $x^2 + y^2 = a^2$, where $\hat{\mathbf{n}}$ is the unit outward normal

   (a) *directly.*

   (b) *by converting to cylindrical coordinates.*

2. *Consider the surface S given parametrically by*

$$x = 2u + v, \quad y = u^2, \quad z = u - v, \quad \text{where} \quad 0 \le u \le 1 \quad \text{and} \quad 0 \le v \le 1.$$

Evaluate the flux of the field $\mathbf{F} = x\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}$ across this surface.

5.5 Volume Integrals

Consider a scalar field f defined within and on the boundary of a region V. we define the **volume integral** of f over V, if it exists, to be

$$\int\int\int_V f(x,y,z)dV = \int\int\int_V f(x,y,z). \tag{5.30}$$

In the general curvilinear coordinates (u_1, u_2, u_3) the element of volume dV is expressed as

$$dV = ds_1 ds_2 ds_3. \tag{5.31}$$

Thus

$$dV = (h_1 du_1)(h_2 du_2)(h_3 du_3). \tag{5.32}$$

Thus, in the Cartesian coodinate system

$$dV = dxdydz; \tag{5.33}$$

in cylindrical coordinates

$$dV = rdrd\theta dz; \tag{5.34}$$

and in spherical coordinates

$$dV = r^2 \sin\phi drd\phi d\theta. \tag{5.35}$$

Example 5.5.1. *If*

$$f(x,y,z) = 4x + xz,$$

evaluate $\int\int\int_V fdV$ *over the rectangular solid bounded by*

$$x = 0, \quad x = 2, \quad y = 0, \quad y = 2, \quad z = 0, \quad z = \frac{3}{2}.$$

Solution 5.5.1.

$$\int\int\int_V fdV = \int\int\int_V (4x + xz)dxdydz \quad (any order of integration is correct).$$

Now

$$\int\int\int_V (4x + xz)dxdydz = \int_0^{3/2}\int_0^1\int_0^2 (4x + xz)dxdydz$$
$$= \int_0^{3/2}\int_0^1 [2x^2 + \frac{1}{2}x^2 z]_0^2 dydz$$

$$
\begin{aligned}
&= \int_0^{3/2}\int_0^1 (8+2z)dydz \\
&= \int_0^{3/2} [8y+2zy]_0^1 dz \\
&= \int_0^{3/2} (8+2z)dz \\
&= \frac{57}{4}.
\end{aligned}
$$

If $f=1$, the$\int\int\int_V$ gives the volume of the region V.

Example 5.5.2. *By evaluating a volume integral of the form*

$$\int\int\int_V$$

deduce the volume of a sphere V of radius ρ.

Solution 5.5.2.

$$Volume \quad = \int\int\int_V dV.$$

Now if V is a sphere of radius ρ, working with spherical coordinates (r,ϕ,θ) we have

$$dV = r^2 \sin\phi dr d\phi d\theta \quad \text{where} \quad 0 \le r \le \rho, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

Hence

$$
\begin{aligned}
Volume \quad &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\rho} r^2 \sin\phi dr d\phi d\theta \\
&= \frac{4}{3}\pi\rho^3.
\end{aligned}
$$

Exercise 5.5.1. *Deduce the volume of a cylinder V of cross-sectional radius rho and length l, by evaluating a suitable volume integral.*

It is often helpful to sketch the region of integration first, then use it to determine the limits and a "convenient" order of integration.

Example 5.5.3. *Sketch the region whose volume is represented by the triple integral*

$$\int_0^2\int_0^3\int_0^{\sqrt{9-y^2}} dxdydz.$$

Exercise 5.5.2. *Find the volume integral of $f(x,y,z) = x+yz$ over the box bounded by the coordinate planes, $x=1$, $y=2$ and $z=1+x$.*

Chapter 6

INTEGRAL THEOREMS

This chapter covers three remarkable theorems which are central to the study of vector calculus, and these are Green's theorem, Stokes' theorem and the divergence theorem.

6.1 Green's Theorem

The theorem converts the line integral of a vector field around a simple closed curve in a domain of the x, y-plane into a double integral over the domain enclosed by the curve.

Theorem 6.1.1. Green's Theorem*: Let D be a domain in the x, y-plane bounded by a piecewise smooth simple closed curve C. If $F_1(x, y)$, $F_2(x, y)$ and their first partial derivatives are continuous in D and on C, then*

$$\oint_C F_1 dx + F_2 dy = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \tag{6.1}$$

where C is taken in the positive (counterclockwise) direction.

Example 6.1.1. *Verify Green's theorem in the plane for*

$$\oint_C (xy + y^2) dx + x^2 dy$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution 6.1.1. *Green's theorem:*

$$\oint_C F_1 dx + F_2 dy = \int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy.$$

For the line integral (left-hand side) $\oint_C F_1 dx + F_2 dy$, *the closed curve* C *is consists of two smooth curves* $C_1: \quad y = x^2$, from $x = 0$ to $x = 1$ *and* $C_2: \quad y = x$, from $x = 1$ back to $x = 0$; *and* $F_1 = xy + y^2$, $F_2 = x^2$.
Along $C_1: \quad y = x^2$ *the line integral is*

$$\int_{C_1} F_1 dx + F_2 dy = \int_{C_1} [x(x^2) + x^4] dx +)x^2)(2xdx)$$
$$= \int_0^1 (3x^3 + x^4) dx = \frac{19}{20}.$$

Along $C_2: \quad y = x$ *the line integral is*

$$\int_{C_2} F_1 dx + F_2 dy = \int_{C_2} [(x)(x) + x^2] dx + x^2 dx$$
$$= \int_1^0 3x^2 = -1.$$

Hence

$$\oint_C F_1 dx + F_2 dy = \int_{C_1} F_1 dx + F_2 dy + \int_{C_2} F_1 dx + F_2 dy = -\frac{1}{20}.$$

For the surface integral (right-hand side) $\int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$, D *is the region between the curves* $y = x^2$ *and* $y = x$. *Thus*

$$\int\int_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \int_{x=0}^1 \int_{y=x^2}^x \left[\frac{\partial (x^2}{\partial x} - \frac{\partial (xy + y^2}{\partial y}\right] dydx$$
$$= \int_{x=0}^1 \int_{y=x^2}^x (x - 2y) dydx$$
$$= \int_{x=0}^1 [xy - y^2]_0^1 dx$$
$$= \int_{x=0}^1 (x^4 - x^3) dx$$
$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = -\frac{1}{20},$$

thus verifying Green's theorem.

Exercise 6.1.1. *1. Use Green's theorem to evaluate*

(a)

$$\oint_C (x+y)dx - 2xdy$$

where C is the unit circle $x^2+y^2=1$, traced in the counterclockwise direction.

(b)

$$\oint_C (x+2y)dx + xydy$$

where C is the ellipse $x^2+4y^2=4$, traced in the counterclockwise direction.

(c)

$$\oint_C (x+y^2)dx - xydy$$

where C is the sides of the square with vertices at $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$, traced in the counterclockwise direction.

(d)

$$\oint_C (x\cos x - e^y)dx - (y^2+xe^y)dy,$$

where C is any piecewise smooth simple closed curve.

2. Verify Green's theorem for the line integral

$$\oint_C 2xy^3dx + 4x^2y^2dy,$$

where C is the boundary of the region in the first quadrant bounded by $x=1$, $y=x^3$ and the x-axis.

6.2 Stokes' Theorem

Observe that Green's theorem in the plane can be written in the form

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_D (\nabla\times\mathbf{F})\cdot\hat{\mathbf{k}}dxdy,$$

where $\mathbf{F}(x,y) = F_1(x,y)\hat{\mathbf{i}} + F_2(x,y)\hat{\mathbf{j}}$ and D is a plane domain bounded by a simple closed curve C. The extension of Green's theorem to three dimensional vector fields leads to what is known as Stokes' theorem.

Theorem 6.2.1. ***Stokes' Theorem****: Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve C. If $\mathbf{F}(x,y,z) = F_1(x,y,z)\hat{\mathbf{i}} + F_2(x,y,z)\hat{\mathbf{j}} + F_3(x,y,z)\hat{\mathbf{k}}$ is continuously differentiable in a domain containing S and C, then*

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}dS, \tag{6.2}$$

where C is traversed in the positive (counterclockwise) direction and $\hat{\mathbf{n}}$ is the positive normal vector to S, determined by the orientation of S according to the right-hand rule.

In other words, Stokes' theorem says that the surface integral of the normal component of the curl of a vector field $\mathbf{F}$, taken over a bounded surface S, equals the line integral of the filed, taken over the closed curve C bounding the surface.

Example 6.2.1. *By means of Stokes' theorem, evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around the ellipse $C: \quad x^2+y^2 = 1, \quad z = y$, where*

$$\mathbf{F} = x\hat{\mathbf{i}} + (x+y)\hat{\mathbf{j}} + (x+y+z)\hat{\mathbf{k}}.$$

Solution 6.2.1. *According to Stokes' theorem*

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}dS,$$

where S is the region bounded by (and including) the ellipse $C: \quad x^2+y^2=1, \quad z=y$.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x+y & x+y+z \end{vmatrix}$$

$$= \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

In parametric form (polar coordinates, r and θ),

$$S: \quad x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = y = r\sin\theta, \qquad 0\le r\le 1, \quad 0\le\theta\le 2\pi.$$

Hence,

$$\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta} = (\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}} + \sin\theta\hat{\mathbf{k}})\times(-r\sin\theta\hat{\mathbf{i}} + r\cos\theta\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}})$$

$$= -r\hat{\mathbf{j}} + r\hat{\mathbf{k}}.$$

Therefore

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int\int_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}dS$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (-r\hat{\mathbf{j}} + r\hat{\mathbf{k}}) dr d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 2r dr d\theta = 2\pi \quad \text{or} \quad -2\pi,$$

depending on the direction.

Exercise 6.2.1. *Use Stokes' theorem to evaluate*

1. $\oint_C (2xy^2 + \sin z)dx + 2x^2 y dy + x \cos z dz$, *around the curve*

   $$C: \quad x = \cos t, \quad y = \sin t, \quad z = \sin t, \quad 0 \leq t \leq 2\pi,$$

   directed with increasing t.

2. $\oint_C (3x + 4y)dx + (2x + 3y^2)dy$ *around the circle* $C: \quad x^2 y^2 = 4$.

6.3 The Divergence Theorem

The divergence theorem, also called Gauss' theorem, establishes an important relationship between an integral over a volume to an integral over the surface which binds the volume.

Theorem 6.3.1. The Divergence Theorem *If* V *is the volume bounded by a closed surface* S *and* $\mathbf{F}$ *is a continuously defferentiable vector field in a domain containing* S*, then*

$$\int \int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \int \int \int_V \nabla \cdot \mathbf{F} dV, \tag{6.3}$$

where $\hat{\mathbf{n}}$ *is the positive unit vector normal to* S.

Example 6.3.1. *Evaluate, via the divergence theorem,* $\int \int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$*, where*

$$\mathbf{F} = 4xz\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$$

and S *is the surface of the cube bounded by*

$$x = 0, \quad x = 1, \quad y = 0, \quad y = 1, \quad z = 0, \quad z = 1.$$

Solution 6.3.1.

$$\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \int\int\int_V \nabla\cdot\mathbf{F}dV$$

$$= \int\int\int_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right]dV$$

$$= \int_0^1\int_0^1\int_0^1 (4z-y)dV = \frac{3}{2}.$$

Exercise 6.3.1. *Verify the divergence theorem for* $\mathbf{F} = 2x^2\hat{\mathbf{i}} - 3y\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$, *where* S *is the cylinder*

$$x^2 + y^2 = 9, \quad z = 0, \quad z = 2.$$